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On some inequalities for relative semi-convex functions

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Abstract
We consider and study a new class of convex functions that are called relative semi-convex functions. Some Hermite-Hadamard inequalities for the relative semi-convex function and its variant forms are derived. Several special cases are also discussed. Results proved in this paper may stimulate further research in this area.

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1 Introduction

Convexity plays a central and fundamental role in the fields of mathematical finance, economics, engineering, management sciences, and optimization theory. In recent years, the concept of convexity has been extended and generalized in several directions using the novel and innovative ideas; see, for example, [1–13] and the references therein. A significant generalization of a convex set and a convex function was the introduction of a relative convex (g-convex) set and a relative convex (g-convex) function by Youness [13]. Noor [14] showed that the optimality condition for a relative convex function on the relative convex set can be characterized by a class of variational inequalities known as general variational inequalities. Motivated by the work of Youness [13] and Noor [14], Chen [2] introduced and studied a new class of functions called relative semi-convex functions. Noor *et al.* [15] derived Hermite-Hadamard inequalities for differentiable relative semi-convex functions. For useful details on Hermite-Hadamard inequalities, see [1, 5–8, 10, 15–22].

Niculescu [7] introduced the concept of relative convexity and proved various properties and generalizations of classical results for relative convexity. Mercer [6] has also proved some useful results for relative convexity.

In this paper, we derive some Hermite-Hadamard inequalities for the relative semi-convex function and the logarithmic relative semi-convex function. The ideas of this paper may stimulate further research in this area.

2 Preliminaries

In this section, we recall some basic results and concepts, which are useful in proving our results. Let $\mathbb{R}^n$ be a finite dimensional space, the inner product of which is denoted by $\langle \cdot, \cdot \rangle$.

Definition 2.1 [13] A set $M \subseteq \mathbb{R}^n$ is said to be a relative convex (g-convex) set if and only if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(1-t)g(x) + tg(y) \in M, \quad \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M, t \in [0,1]. \tag{2.1}$$

It is known [23] that if M is a relative convex set, then it may not be a classical convex set. For example, for $M = [-1, -\frac{1}{2}] \cup [0,1]$ and $g(x) = x^2$, $\forall x \in \mathbb{R}$. Clearly, this is a relative convex set but not a classical convex set.

Definition 2.2 [13] A function f is said to be a relative convex (g-convex) function on the relative convex set M if and only if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} &f\bigl((1-t)g(x) + tg(y)\bigr) \le (1-t)f\bigl(g(x)\bigr) + tf\bigl(g(y)\bigr), \\ &\quad \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M, t \in [0,1]. \end{aligned} \tag{2.2}$$

Every convex function f on a convex set is a relative convex function. However, the converse is not true. There are functions which are relative convex functions but may not be convex functions in the classical sense, see [13].

Definition 2.3 [2] A function f is said to be a relative semi-convex function if and only if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f\bigl((1-t)g(x) + tg(y)\bigr) \le (1-t)f(x) + tf(y), \quad x, y \in M, t \in [0,1]. \tag{2.3}$$

Remark 2.1 A relative semi-convex function on a relative convex set is not necessarily a relative convex function, see [2].

Definition 2.4 [3] A function $f : M \to \mathbb{R}^+$ is said to be relative logarithmic semi-convex on a relative convex set M if

$$f\bigl((1-t)g(x) + tg(y)\bigr) \le \bigl[f(x)\bigr]^{1-t}\bigl[f(y)\bigr]^t, \quad \forall x, y \in M, t \in [0,1]. \tag{2.4}$$

From Definition 2.4 it follows that

$$\begin{aligned} f\bigl((1-t)g(x) + tg(y)\bigr) &\le \bigl[f(x)\bigr]^{1-t}\bigl[f(y)\bigr]^t \\ &\le (1-t)f(x) + tf(y), \end{aligned}$$

which shows that every relative logarithmic semi-convex function is a relative semi-convex function, but the converse is not true.

Definition 2.5 [24] Let $f \in L_1[a,b]$. The generalized Riemann-Liouville fractional integrals $J^\alpha_{a^+}f$ and $J^\alpha_{b^-}f$ of order $\alpha > 0$ with $p \ge 0$ are defined by

$$J^\alpha_{p,a^+}f(x) = \frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \bigl(x^{p+1} - t^{p+1}\bigr)^{\alpha-1} t^p f(t)\,dt, \quad x > a,$$

and

$$J^{\alpha}_{p,b^-}f(x) = \frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \left(t^{p+1} - x^{p+1}\right)^{\alpha-1} t^p f(t)\,dt, \quad x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}x^{\alpha-1}\,dx$ is the gamma function.

If $p = 0$, then Definition 2.5 reduces to the definition for classical Riemann-Liouville integrals. See also [25, 26].

Definition 2.6 [20] Two functions f and g are said to be similarly ordered on $I \subseteq \mathbb{R}$ if

$$\langle f(x) - f(y), g(x) - g(y)\rangle \geq 0, \quad \forall x, y \in I.$$

Let $M = I = [g(a), g(b)]$ be a relative semi-convex set. We now define a relative semi-convex function on I, which appears to be a new one.

Definition 2.7 Let $I = [g(a), g(b)]$, then f is called a relative semi-convex function if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ g(a) & g(x) & g(b) \\ f(a) & f(g(x)) & f(b) \end{vmatrix} \geq 0; \quad g(a) \leq g(x) \leq g(b).$$

One can easily show that the following are equivalent:

1. f is a relative semi-convex function on a relative convex set.
2. $f(g(x)) \leq f(a) + \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a))$.
3. $\frac{f(g(x))-f(a)}{g(x)-g(a)} \leq \frac{f(b)-f(a)}{g(b)-g(a)} \leq \frac{f(b)-f(g(x))}{g(b)-g(x)}$.
4. $\frac{-f(a)}{(g(x)-g(a))(g(b)-g(a))} + \frac{f(g(x))}{(g(b)-g(x))(g(x)-g(a))} - \frac{f(b)}{(g(b)-g(a))(g(b)-g(x))} \geq 0$.
5. $(g(b) - g(x))f(a) - (g(b) - g(a))f(g(x)) + (g(x) - g(a))f(b) \geq 0$,

where $g(x) = (1-t)g(a) + tg(b) \in M$, $t \in [0, 1]$.

For the applications of the relative convex functions, see [27].

Remark 2.2 We note that if f is a differentiable relative semi-convex function, then

$$f\big(g(y)\big) - f(x) \geq \left\langle \frac{f'(x)}{g'(x)}, g(y) - g(x) \right\rangle, \quad \forall g(y) \in \big(g(a), g(b)\big).$$

3 Main results

In this section we discuss our main results.

Essentially using the techniques of [7], one can prove the following results for relative semi-convexity.

Lemma 3.1 *Let f be a relative semi-convex function. If g is not a constant function, then*

$$g(a) = g(x) \quad \textit{implies} \quad f(a) = f\big(g(x)\big).$$

Lemma 3.2 *Let $f : I \to \mathbb{R}$ be a relative semi-convex function, where $I = [g(a), g(b)]$. If $g(x) \notin \{g(a), g(b)\}$, then*

$$\frac{f(b) - f(g(x))}{g(b) - g(x)} \geq \frac{f(a) - f(g(x))}{g(a) - g(x)}.$$

Lemma 3.3 *Let f be a relative semi-convex function. Consider $g(x_1), g(x_2), \ldots, g(x_n) \in I$, $g(y_1), g(y_2), \ldots, g(y_n) \in I$ and weights $\omega_1, \omega_2, \ldots, \omega_n \in \mathbb{R}$ such that:*

(i) $g(x_1) \geq g(x_2) \geq \cdots \geq g(x_n)$ *and* $g(y_1) \geq g(y_2) \geq \cdots \geq g(y_n)$,
(ii) $\sum_{k=1}^{r} \omega_k g(x_k) \leq \sum_{k=1}^{r} \omega_k g(y_k)$, $\forall r = 1, \ldots, n$,
(iii) $\sum_{k=1}^{n} \omega_k g(x_k) = \sum_{k=1}^{n} \omega_k g(y_k)$,

then we have

$$\sum_{k=1}^{n} \omega_k f\bigl(g(x_k)\bigr) \leq \sum_{k=1}^{n} \omega_k f(y_k).$$

Lemma 3.4 *Let f be a relative semi-convex function. Consider $g(x_1), g(x_2), \ldots, g(x_n) \in I$, $g(y_1), g(y_2), \ldots, g(y_n) \in I$ and weights $\omega_1, \omega_2, \ldots, \omega_n \in \mathbb{R}$ such that*

(i) $g(x_1) \geq g(x_2) \geq \cdots \geq g(x_n)$ *and* $g(y_1) \geq g(y_2) \geq \cdots \geq g(y_n)$,
(ii) $\sum_{k=1}^{r} \omega_k g(x_k) \leq \sum_{k=1}^{r} \omega_k g(y_k)$, $\forall r = 1, \ldots, n$,
(iii) $\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0$,

then we have

$$\sum_{k=1}^{n} \omega_k f\bigl(g(x_k)\bigr) \leq \sum_{k=1}^{n} \omega_k f(y_k).$$

Lemma 3.5 *Let f be a relative semi-convex function, then, for all $g(a) < g(c) < g(d) < g(b)$, we have*

$$\frac{f(a) + f(b)}{2} - f\left(\frac{g(a) + g(b)}{2}\right) \geq \frac{f(c) + f(d)}{2} - f\left(\frac{g(c) + g(d)}{2}\right).$$

Theorem 3.6 *Let f and w be two relative semi-convex functions. Then the product of f and w will be a relative semi-convex function iff f and w are similarly ordered functions.*

Proof Since f and w are relative semi-convex functions, so we have

$$\begin{aligned}
& f\bigl((1-t)g(a) + tg(b)\bigr) w\bigl((1-t)g(a) + tg(b)\bigr) \\
&\quad \leq \bigl[(1-t)f(a) + tf(b)\bigr]\bigl[(1-t)w(a) + tw(b)\bigr] \\
&\quad = [1-t]^2 f(a)w(a) + t(1-t)f(a)w(b) + t(1-t)f(b)w(a) + [t]^2 f(b)w(b) \\
&\quad = (1-t)f(a)w(a) + tf(b)w(b) - t(1-t)\bigl[f(a)w(a) + f(b)w(b) - f(b)w(a) - f(a)w(b)\bigr] \\
&\quad \leq (1-t)f(a)w(a) + tf(b)w(b),
\end{aligned}$$

where we have used the fact that f and w are similarly ordered. This completes the proof. □

We now obtain some Hermite-Hadamard inequalities for relative semi-convex functions.

Theorem 3.7 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a relative semi-convex function on $I = [g(a), g(b)]$ with $g(a) < g(b)$, then we have*

$$f\left(\frac{g(a)+g(b)}{2}\right) \le \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f\big(g(x)\big)\,dg(x) \le \frac{f(a)+f(b)}{2}. \tag{3.1}$$

Proof Let f be relative semi-convex. Then

$$\begin{aligned}
f\left(\frac{g(a)+g(b)}{2}\right) &= \int_0^1 f\left(\frac{g(a)+g(b)}{2}\right) dt \\
&= \int_0^1 f\left(\frac{(1-t)g(a)+tg(b)+tg(a)+(1-t)g(b)}{2}\right) dt \\
&\le \frac{1}{2}\int_0^1 \big[f\big((1-t)g(a)+tg(b)\big)+f\big(tg(a)+(1-t)g(b)\big)\big]\,dt \\
&= \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f\big(g(x)\big)\,dg(x) = \int_0^1 f\big((1-t)g(a)+tg(b)\big)\,dt \\
&\le \int_0^1 \big((1-t)f(a)+tf(b)\big)\,dt = \frac{f(a)+f(b)}{2}.
\end{aligned}$$

□

Using the technique of [21], we can prove the following result.

Lemma 3.8 *Let f be a semi-relative convex function. Then, for any $g(x) \in [g(a), g(b)]$, we have*

$$f\big(g(a)+g(b)-g(x)\big) \le f(a)+f(b)-f\big(g(x)\big).$$

Theorem 3.9 *Let f be a relative semi-convex function and let $w : [g(a), g(b)] \to \mathbb{R}$ be nonnegative, integrable and symmetric about $\frac{g(a)+g(b)}{2}$. Then*

$$\begin{aligned}
f\left(\frac{g(a)+g(b)}{2}\right) \int_{g(a)}^{g(b)} w\big(g(x)\big)\,dg(x) &\le \int_{g(a)}^{g(b)} f\big(g(x)\big)w\big(g(x)\big)\,dg(x) \\
&\le \frac{f(a)+f(b)}{2} \int_{g(a)}^{g(b)} w\big(g(x)\big)\,dg(x).
\end{aligned} \tag{3.2}$$

Proof Since f is a relative semi-convex function and $w : [g(a), g(b)] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{g(a)+g(b)}{2}$, we have

$$\begin{aligned}
&f\left(\frac{g(a)+g(b)}{2}\right) \int_{g(a)}^{g(b)} w\big(g(x)\big)\,dg(x) \\
&\quad= \int_{g(a)}^{g(b)} f\left(\frac{g(a)+g(b)}{2}\right) w\big(g(x)\big)\,dg(x) \\
&\quad\le \int_{g(a)}^{g(b)} \left[\frac{1}{2}\Big(f\big(g(a)+g(b)-g(x)\big)+f\big(g(x)\big)\Big)\right] w\big(g(x)\big)\,dg(x) \\
&\quad\le \int_{g(a)}^{g(b)} f\big(g(x)\big)w\big(g(x)\big)\,dg(x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{g(a)}^{g(b)} f\bigl(g(a)+g(b)-g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x) + \frac{1}{2}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x)\\
&\le \frac{1}{2}\int_{g(a)}^{g(b)} \bigl\{f(a)+f(b)-f\bigl(g(x)\bigr)\bigr\}w\bigl(g(x)\bigr)\,dg(x) + \frac{1}{2}\int_{a}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x)\\
&= \frac{f(a)+f(b)}{2}\int_{g(a)}^{g(b)} w\bigl(g(x)\bigr)\,dg(x).
\end{aligned}
$$

This completes the proof. □

Theorem 3.10 *Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be relative semi-convex functions on I with $g(a) < g(b)$. Then, for all $t \in [0,1]$, we have*

$$
\begin{aligned}
&2f\left(\frac{g(a)+g(b)}{2}\right)w\left(\frac{g(a)+g(b)}{2}\right) - \left[\frac{1}{6}M(a,b) + \frac{1}{2}N(a,b)\right]\\
&\quad\le \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x) \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),
\end{aligned}
$$

where

$$M(a,b) = f(a)w(a) + f(b)w(b), \tag{3.3}$$

$$N(a,b) = f(a)w(b) + f(b)w(a). \tag{3.4}$$

Proof Let f and w be relative semi-convex functions. Then

$$
\begin{aligned}
&f\left(\frac{g(a)+g(b)}{2}\right)w\left(\frac{g(a)+g(b)}{2}\right)\\
&\quad= f\left(\frac{tg(a)+(1-t)g(b)+(1-t)g(a)+tg(b)}{2}\right)\\
&\qquad\times w\left(\frac{tg(a)+(1-t)g(b)+(1-t)g(a)+tg(b)}{2}\right)\\
&\quad\le \frac{1}{2}\bigl[f\bigl(tg(a)+(1-t)g(b)\bigr)+f\bigl((1-t)g(a)+tg(b)\bigr)\bigr]\\
&\qquad\times \frac{1}{2}\bigl[w\bigl(tg(a)+(1-t)g(b)\bigr)+w\bigl((1-t)g(a)+tg(b)\bigr)\bigr]\\
&\quad= \frac{1}{4}\bigl[f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl(tg(a)+(1-t)g(b)\bigr)\\
&\qquad+f\bigl((1-t)g(a)+tg(b)\bigr)w\bigl((1-t)g(a)+tg(b)\bigr)\bigr]\\
&\qquad+\frac{1}{4}\bigl[f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl((1-t)g(a)+tg(b)\bigr)\\
&\qquad+f\bigl((1-t)g(a)+tg(b)\bigr)w\bigl(tg(a)+(1-t)g(b)\bigr)\bigr]\\
&\quad\le \frac{1}{4}\bigl[f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl(tg(a)+(1-t)g(b)\bigr)\\
&\qquad+f\bigl((1-t)g(a)+tg(b)\bigr)w\bigl((1-t)g(a)+tg(b)\bigr)\bigr]\\
&\qquad+\frac{1}{4}\bigl[2t(1-t)\bigl(f(a)w(a)+f(b)w(b)\bigr)+\bigl(t^2+(1-t)^2\bigr)\bigl(f(b)w(a)+f(a)w(b)\bigr)\bigr].
\end{aligned}
$$

Integrating with respect to t on $[0,1]$, we have

$$\begin{aligned} &f\left(\frac{g(a)+g(b)}{2}\right)w\left(\frac{g(a)+g(b)}{2}\right) \\ &\quad\le \frac{1}{4}\left[\frac{2}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x)\right] + \frac{1}{2}\left[\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)\right]. \end{aligned}$$

This implies that

$$\begin{aligned} &2f\left(\frac{g(a)+g(b)}{2}\right)w\left(\frac{g(a)+g(b)}{2}\right) - \left[\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)\right] \\ &\quad\le \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x) \\ &\quad= \int_0^1 f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl(tg(a)+(1-t)g(b)\bigr)\,dt \\ &\quad\le \int_0^1 \bigl[tf(a)+(1-t)f(b)\bigr]\bigl[tw(a)+(1-t)w(b)\bigr]\,dt \\ &\quad= \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b). \end{aligned}$$

This completes the proof. □

Theorem 3.11 *Let $f,w : I \subseteq \mathbb{R} \to \mathbb{R}$ be relative semi-convex functions on I with $g(a) < g(b)$. If w is symmetric about $\frac{g(a)+g(b)}{2}$, then for all $t \in [0,1]$ we have*

$$\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(a)+g(b)-g(x)\bigr)\,dg(x) \le \frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),$$

where $M(a,b)$ and $N(a,b)$ are given by (3.3) and (3.4), $\Theta(a,b) = [f(a)]^2 + [f(b)]^2 + [w(a)]^2 + [w(b)]^2$.

Proof Since f and w are relative semi-convex functions, then we have

$$\begin{aligned} &\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(a)+g(b)-g(x)\bigr)\,dg(x) \\ &\quad= \int_0^1 f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl((1-t)g(a)+tg(b)\bigr)\,dt \\ &\quad\le \frac{1}{2}\int_0^1 \bigl\{\bigl[f\bigl(tg(a)+(1-t)g(b)\bigr)\bigr]^2 + \bigl[w\bigl((1-t)g(a)+tg(b)\bigr)\bigr]^2\bigr\}\,dt \\ &\quad\le \frac{1}{2}\int_0^1 \bigl\{\bigl[tf(a)+(1-t)f(b)\bigr]^2 + \bigl[(1-t)w(a)+tw(b)\bigr]^2\bigr\}\,dt \\ &\quad= \frac{1}{6}\bigl\{\bigl[f(a)\bigr]^2+\bigl[f(b)\bigr]^2+f(a)f(b)+\bigl[w(a)\bigr]^2+\bigl[w(b)\bigr]^2+w(a)w(b)\bigr\} \\ &\quad\le \frac{1}{4}\bigl\{\bigl[f(a)\bigr]^2+\bigl[f(b)\bigr]^2+\bigl[w(a)\bigr]^2+\bigl[w(b)\bigr]^2\bigr\} = \frac{1}{4}\Theta(a,b) \\ &\quad\le \int_0^1 \bigl(tf(a)+(1-t)f(b)\bigr)\bigl((1-t)w(a)+tw(b)\bigr)\,dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6} f(a)w(a) + \frac{1}{3} f(a)w(b) + \frac{1}{3} f(b)w(a) + \frac{1}{6} f(b)w(b) \\
&= \frac{1}{6} M(a,b) + \frac{1}{3} N(a,b).
\end{aligned}
$$

The desired result. □

Theorem 3.12 *Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be similarly ordered and relative semi-convex functions on I with $g(a) < g(b)$. Then, for all $t \in [0,1]$, we have*

$$
\begin{aligned}
&2f\left(\frac{g(a)+g(b)}{2}\right) w\left(\frac{g(a)+g(b)}{2}\right) - \frac{1}{4} M(a,b) \\
&\quad \le \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f\big(g(x)\big) w\big(g(x)\big)\, dg(x) \le \frac{f(a)w(a)+f(b)w(b)}{2},
\end{aligned}
$$

where $M(a,b)$ is given by (3.3).

Proof Since f and w are similarly ordered functions, the proof follows from Theorem 3.10. □

Theorem 3.13 *Let f be a relative semi-convex function, then for all $\lambda \in (0,1)$ we have*

$$
\begin{aligned}
f\left(\frac{g(a)+g(b)}{2}\right) &\le \Delta_1(\lambda) \le \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f\big(g(x)\big)\, dg(x) \\
&\le \Delta_2(\lambda) \le \frac{f(a)+f(b)}{2},
\end{aligned}
\tag{3.5}
$$

where

$$
\Delta_1(\lambda) = \lambda f\left(\frac{(2-\lambda)g(a)+\lambda g(b)}{2}\right) + (1-\lambda) f\left(\frac{(1-\lambda)g(a)+(1+\lambda)g(b)}{2}\right)
$$

and

$$
\Delta_2(\lambda) = \frac{f((1-\lambda)g(a)+\lambda g(b)) + \lambda f(a) + (1-\lambda) f(b)}{2}.
$$

Proof We divide the interval $[g(a), g(b)]$ into $[g(a), (1-\lambda)g(a)+\lambda g(b)]$ and $[(1-\lambda)g(a)+\lambda g(b), g(b)]$. Using the left-hand side of (3.1), we have

$$
f\left(\frac{(2-\lambda)g(a)+\lambda g(b)}{2}\right) \le \frac{1}{\lambda(g(b)-g(a))} \int_{g(a)}^{(1-\lambda)g(a)+\lambda g(b)} f\big(g(x)\big)\, dg(x), \tag{3.6}
$$

$$
f\left(\frac{(1-\lambda)g(a)+(1+\lambda g(b))}{2}\right) \le \frac{1}{(1-\lambda)(g(b)-g(a))} \int_{(1-\lambda)g(a)+\lambda g(b)}^{g(b)} f\big(g(x)\big)\, dg(x). \tag{3.7}
$$

Multiplying (3.6) by λ and (3.7) by $(1-\lambda)$, and then adding the resultant, we have

$$
\begin{aligned}
\Delta_1(\lambda) &= \lambda f\left(\frac{(2-\lambda)g(a)+\lambda g(b)}{2}\right) + (1-\lambda) f\left(\frac{(1-\lambda)g(a)+(1+\lambda g(b))}{2}\right) \\
&\le \frac{1}{g(b)-g(a)} \int_{g(a)}^{g(b)} f\big(g(x)\big)\, dg(x).
\end{aligned}
\tag{3.8}
$$

Now, using the right-hand side of (3.1), we have

$$\begin{aligned}\frac{1}{\lambda(g(b)-g(a))}\int_{g(a)}^{(1-\lambda)g(a)+\lambda g(b)} f\big(g(x)\big)\,dg(x) &\le \frac{f(g(a))+f((1-\lambda)g(a)+\lambda g(b))}{2}\\ &\le \frac{f(a)+f((1-\lambda)g(a)+\lambda g(b))}{2}, \end{aligned}\tag{3.9}$$

$$\begin{aligned}\frac{1}{(1-\lambda)(g(b)-g(a))}\int_{(1-\lambda)g(a)+\lambda g(b)}^{g(b)} f\big(g(x)\big)\,dg(x) &\le \frac{f((1-\lambda)g(a)+\lambda g(b))+f(g(b))}{2}\\ &\le \frac{f((1-\lambda)g(a)+\lambda g(b))+f(b)}{2}. \end{aligned}\tag{3.10}$$

Multiplying (3.9) by λ and (3.10) by $(1-\lambda)$ and adding the resultant, we have

$$\begin{aligned}\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)\,dg(x) &\le \frac{f((1-\lambda)g(a)+\lambda g(b))+\lambda f(a)+(1-\lambda)f(b)}{2}\\ &= \Delta_2(\lambda). \end{aligned}\tag{3.11}$$

Now, using the fact that f is a relative semi-convex function, and also every convex function is a relative semi-convex function, we have

$$\begin{aligned} &f\left(\frac{g(a)+g(b)}{2}\right)\\ &\quad= f\left(\lambda\frac{(2-\lambda)g(a)+\lambda g(b)}{2}+(1-\lambda)\frac{(1-\lambda)g(a)+(1+\lambda)g(b)}{2}\right)\\ &\quad\le \lambda f\left(\frac{(2-\lambda)g(a)+\lambda g(b)}{2}\right)+(1-\lambda)f\left(\frac{(1-\lambda)g(a)+(1+\lambda)g(b)}{2}\right)=\Delta_1(\lambda)\\ &\quad\le \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)\,dg(x)\\ &\quad\le \frac{1}{2}\big[\lambda f\big((1-\lambda)g(a)+\lambda g(b)\big)+\lambda f(a)+(1-\lambda)f\big((1-\lambda)g(a)+\lambda g(b)\big)+(1-\lambda)f(b)\big]\\ &\quad= \frac{1}{2}\big[f\big((1-\lambda)g(a)+\lambda g(b)\big)+\lambda f(a)+(1-\lambda)f(b)\big]=\Delta_2(\lambda)\\ &\quad\le \frac{1}{2}\big[(1-\lambda)f(a)+\lambda f(b)+\lambda f(a)+(1-\lambda)f(b)\big]=\frac{f(a)+f(b)}{2}, \end{aligned}\tag{3.12}$$

the required result. □

Remark 3.1 For suitable and different choices of $\lambda \in (0,1)$ and $g = I$ in Theorem 3.13, one can obtain several new and previously known results for various classes of convex functions.

We now prove the Hermite-Hadamard type inequalities for relative semi-convex functions via fractional integrals.

Theorem 3.14 *Let f be a relative semi-convex function. Then*

$$J^{\alpha}_{p,g(a)^+}f\big(g(b)\big)+J^{\alpha}_{p,g(b)^-}f\big(g(a)\big)\le \big[f(a)+f(b)\big]\big[J^{\alpha}_{p,g(a)^+}(1)+J^{\alpha}_{p,g(b)^-}(1)\big],\quad \alpha>0, p\ge 0.$$

Proof Since f is a relative semi-convex function on M, so

$$\begin{aligned}
&\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[(1-t)g(a)+tg(b)\bigr]^{p+1}\bigr)^{\alpha-1}\\
&\quad\times\bigl[(1-t)g(a)+tg(b)\bigr]^p f\bigl((1-t)g(a)+tg(b)\bigr)\,dt\\
&\quad\le\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}f(a)\int_0^1\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[(1-t)g(a)+tg(b)\bigr]^{p+1}\bigr)^{\alpha-1}\\
&\quad\times\bigl[(1-t)g(a)+tg(b)\bigr]^p(1-t)\,dt\\
&\quad+\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}f(b)\int_0^1\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[(1-t)g(a)+tg(b)\bigr]^{p+1}\bigr)^{\alpha-1}\\
&\quad\times\bigl[(1-t)g(a)+tg(b)\bigr]^p(t)\,dt.
\end{aligned}$$

Let $g(x)=(1-t)g(a)+tg(b)$, then $dt=\frac{dg(x)}{g(b)-g(a)}$. Take $t=\frac{g(x)-g(a)}{g(b)-g(a)}$, $1-t=\frac{g(b)-g(x)}{g(b)-g(a)}$. Then we have

$$\begin{aligned}
&\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)(g(b)-g(a))}\int_{g(a)}^{g(b)}\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[g(x)\bigr]^{p+1}\bigr)^{\alpha-1}\bigl[g(x)\bigr]^p f\bigl(g(x)\bigr)\,dg(x)\\
&\quad\le\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\frac{f(a)}{g(b)-g(a)}\int_{g(a)}^{g(b)}\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[g(x)\bigr]^{p+1}\bigr)^{\alpha-1}\bigl[g(x)\bigr]^p\frac{g(b)-g(x)}{g(b)-g(a)}\,dg(x)\\
&\quad+\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\frac{f(b)}{g(b)-g(a)}\\
&\quad\times\int_{g(a)}^{g(b)}\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[g(x)\bigr]^{p+1}\bigr)^{\alpha-1}\bigl[g(x)\bigr]^p\frac{g(x)-g(a)}{g(b)-g(a)}\,dg(x)\\
&\quad\le\bigl[f(a)+f(b)\bigr]\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{g(a)}^{g(b)}\bigl(\bigl[g(b)\bigr]^{p+1}-\bigl[g(x)\bigr]^{p+1}\bigr)^{\alpha-1}\bigl[g(x)\bigr]^p\,dg(x).
\end{aligned}$$

This implies that

$$J^{\alpha}_{p,g(a)^+}f\bigl(g(b)\bigr)\le\bigl[f(a)+f(b)\bigr]J^{\alpha}_{p,g(a)^+}(1). \tag{3.13}$$

Also

$$\begin{aligned}
&\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\bigl(\bigl[(1-t)g(a)+tg(b)\bigr]^{p+1}-\bigl[g(a)\bigr]^{p+1}\bigr)^{\alpha-1}\\
&\quad\times\bigl[(1-t)g(a)+tg(b)\bigr]^p f\bigl((1-t)g(a)+tg(b)\bigr)\,dt\\
&\quad\le\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}f(a)\int_0^1\bigl(\bigl[(1-t)g(a)+tg(b)\bigr]^{p+1}-\bigl[g(a)\bigr]^{p+1}\bigr)^{\alpha-1}\\
&\quad\times\bigl[(1-t)g(a)+tg(b)\bigr]^p(1-t)\,dt\\
&\quad+\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}f(b)\int_0^1\bigl(\bigl[(1-t)g(a)+tg(b)\bigr]^{p+1}-\bigl[g(a)\bigr]^{p+1}\bigr)^{\alpha-1}\\
&\quad\times\bigl[(1-t)g(a)+tg(b)\bigr]^p(t)\,dt.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)(g(b)-g(a))}\int_{g(a)}^{g(b)}\big(\big[g(x)\big]^{p+1}-\big[g(a)\big]^{p+1}\big)^{\alpha-1}\big[g(x)\big]^{p}f\big(g(x)\big)\,dg(x)\\
&\quad\leq\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\frac{f(a)}{g(b)-g(a)}\int_{g(a)}^{g(b)}\big(\big[g(x)\big]^{p+1}-\big[g(a)\big]^{p+1}\big)^{\alpha-1}\big[g(x)\big]^{p}\frac{g(b)-g(x)}{g(b)-g(a)}\,dg(x)\\
&\qquad+\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\frac{f(b)}{g(b)-g(a)}\int_{g(a)}^{g(b)}\big(\big[g(x)\big]^{p+1}-\big[g(a)\big]^{p+1}\big)^{\alpha-1}\big[g(x)\big]^{p}\frac{g(x)-g(a)}{g(b)-g(a)}\,dg(x)\\
&\quad\leq\big[f(a)+f(b)\big]\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\int_{g(a)}^{g(b)}\big(\big[g(x)\big]^{p+1}-\big[g(a)\big]^{p+1}\big)^{\alpha-1}\big[g(x)\big]^{p}\,dg(x).
\end{aligned}$$

This implies that

$$J^{\alpha}_{p,g(b)^-}f\big(g(a)\big)\leq\big[f(a)+f(b)\big]J^{\alpha}_{p,g(b)^-}(1).\tag{3.14}$$

Combining (3.13) and (3.14), we have the required result. □

Remark 3.2 We can prove the Hermite-Hadamard inequality for the classical Riemann-Liouville integrals as follows:

$$f\left(\frac{g(a)+g(b)}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(g(b)-g(a))^{\alpha}}\big[J^{\alpha}_{g(a)^+}f\big(g(b)\big)+J^{\alpha}_{g(b)^-}f\big(g(a)\big)\big]\leq\frac{f(a)+f(b)}{2}.$$

We now derive the Hermite-Hadamard inequalities for the class of relative logarithmic semi-convex functions.

Theorem 3.15 *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a relative logarithmic semi-convex function, then for all $t\in[0,1]$ we have*

$$f\left(\frac{g(a)+g(b)}{2}\right)\leq\exp\left[\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}\log f\big(g(x)\big)\,dg(x)\right]\leq\sqrt{f(a)f(b)}.$$

Theorem 3.16 *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a relative logarithmic semi-convex function, then for all $t\in[0,1]$,*

$$\begin{aligned}
f\left(\frac{g(a)+g(b)}{2}\right)&\leq\exp\left[\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}\log f\big(g(x)\big)\,dg(x)\right]\\
&\leq\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}G\big(f\big(g(x)\big),f\big(g(a)+g(b)-g(x)\big)\big)\,dg(x)\\
&\leq\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)}f\big(g(x)\big)\,dg(x)\leq L\big[f(b),f(a)\big]\leq\frac{f(a)+f(b)}{2},
\end{aligned}$$

where $L[f(b),f(a)]=\frac{f(b)-f(a)}{\log f(b)-\log f(a)}$, and $G[f(a),f(b)]=\sqrt{f(a)f(b)}$.

Proof The proof of the first inequality follows directly from Theorem 3.15. For the second inequality, we consider

$$\begin{aligned}
&\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} G\big(f\big(g(x)\big), f\big(g(a)+g(b)-g(x)\big)\big)\,dg(x)\\
&\quad=\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} \exp\big[\log G\big(f\big(g(x)\big), f\big(g(a)+g(b)-g(x)\big)\big)\big]\,dg(x)\\
&\quad\geq \exp\bigg[\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} \log G\big(f\big(g(x)\big), f\big(g(a)+g(b)-g(x)\big)\big)\,dg(x)\bigg]\\
&\quad=\exp\bigg[\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} \frac{\log f(g(x))+\log f(g(a)+g(b)-g(x))}{2}\,dg(x)\bigg]\\
&\quad=\exp\bigg[\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} \log f\big(g(x)\big)\,dg(x)\bigg].
\end{aligned}$$

Using the *AM-GM* inequality, we have

$$G\big(f\big(g(x)\big), f\big(g(a)+g(b)-g(x)\big)\big)\leq\frac{f(g(x))+f(g(a)+g(b)-g(x))}{2}.$$

Integrating the above inequality with respect to x on $[g(a), g(b)]$, we have

$$\begin{aligned}
&\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} G\big(f\big(g(x)\big), f\big(g(a)+g(b)-g(x)\big)\big)\,dg(x)\\
&\quad\leq\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)\,dg(x).
\end{aligned}$$

Now, using the fact that f is a relative semi-convex function and applying the change of variable technique on the right-hand side of the above inequality completes the proof. □

Theorem 3.17 *Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be relative logarithmic semi-convex functions, then we have*

$$\begin{aligned}
\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)w\big(g(x)\big)\,dg(x)&\leq L\big[f(a)w(b), f(a)w(a)\big]\\
&\leq\frac{f(a)w(a)+f(b)w(b)}{2}\leq\frac{1}{4}\Theta(a,b),
\end{aligned}$$

where $\Theta(a,b)=[f(a)]^2+[f(b)]^2+[w(a)]^2+[w(b)]^2$.

Proof Let f and w be relative logarithmic semi-convex functions. Then

$$\begin{aligned}
&\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)w\big(g(x)\big)\,dg(x)\\
&\quad=\int_0^1 f\big((1-t)g(a)+tg(b)\big)w\big((1-t)g(a)+tg(b)\big)\\
&\quad\leq\int_0^1\big[f(a)w(a)\big]^{1-t}\big[f(b)w(b)\big]^t\,dt=\frac{f(b)w(b)-f(a)w(a)}{\log f(b)w(b)-\log f(a)w(a)}
\end{aligned}$$

$$
\begin{aligned}
&= L\bigl[f(b)w(b), f(a)w(a)\bigr] \le \frac{f(a)w(a)+f(b)w(b)}{2} \\
&\le \frac{1}{2}\int_0^1 \bigl[\bigl\{f\bigl((1-t)g(a)+tg(b)\bigr)\bigr\}^2 + \bigl\{w\bigl((1-t)g(a)+tg(b)\bigr)\bigr\}^2\bigr]\,dt \\
&\le \frac{1}{2}\int_0^1 \bigl[\bigl\{\bigl[f(a)\bigr]^{1-t}\bigl[f(b)\bigr]^t\bigr\}^2 + \bigl\{\bigl[w(a)\bigr]^{1-t}\bigl[w(b)\bigr]^t\bigr\}^2\bigr]\,dt \\
&= \frac{1}{4}\left[\frac{[f(a)+f(b)][f(b)-f(a)]}{\log f(b)-\log f(a)} + \frac{[w(a)+w(b)][w(b)-w(a)]}{\log w(b)-\log w(a)}\right] \\
&\le \frac{1}{8}\bigl[\bigl[f(a)+f(b)\bigr]^2 + \bigl[w(a)+w(b)\bigr]^2\bigr] \le \frac{1}{4}\Theta(a,b). \qquad \square
\end{aligned}
$$

Theorem 3.18 *Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be relative logarithmic semi-convex functions, then*

$$
\begin{aligned}
&\log w\left(\frac{g(a)+g(b)}{2}\right) - \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} \log w\bigl(g(x)\bigr)\,dg(x) \\
&\quad\le \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} \log f\bigl(g(x)\bigr)\,dg(x) - \log f\left(\frac{g(a)+g(b)}{2}\right).
\end{aligned}
$$

Proof Let f and w be relative logarithmic semi-convex functions. Then

$$
\begin{aligned}
&\log f\left(\frac{g(a)+g(b)}{2}\right) w\left(\frac{g(a)+g(b)}{2}\right) \\
&\quad= \log\left[f\left(\frac{(1-t)g(a)+tg(b)+tg(a)+(1-t)g(b)}{2}\right)\right. \\
&\qquad\left.\times w\left(\frac{(1-t)g(a)+tg(b)+tg(a)+(1-t)g(b)}{2}\right)\right] \\
&\quad\le \log\bigl[\bigl[f\bigl((1-t)g(a)+tg(b)\bigr)f\bigl(tg(a)+(1-t)g(b)\bigr)\bigr]^{\frac{1}{2}} \\
&\qquad\times \bigl[w\bigl((1-t)g(a)+tg(b)\bigr)w\bigl(tg(a)+(1-t)g(b)\bigr)\bigr]^{\frac{1}{2}}\bigr] \\
&\quad= \frac{1}{2}\bigl[\log f\bigl((1-t)g(a)+tg(b)\bigr) + \log f\bigl(tg(a)+(1-t)g(b)\bigr)\bigr] \\
&\qquad+ \frac{1}{2}\bigl[\log w\bigl((1-t)g(a)+tg(b)\bigr) + \log w\bigl(tg(a)+(1-t)g(b)\bigr)\bigr].
\end{aligned}
$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we have the required result. □

Theorem 3.19 *Let $f, w : I \subseteq \mathbb{R} \to \mathbb{R}$ be relative logarithmic semi-convex functions, then*

$$
\begin{aligned}
&\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(a)+g(b)-g(x)\bigr)\,dg(x) \\
&\quad\le \frac{f(a)w(b)-f(b)w(a)}{\log f(a)w(b)-\log f(b)w(a)} \le \frac{1}{4}\Theta(a,b),
\end{aligned}
$$

where $\Theta(a,b) = [f(a)]^2 + [f(b)]^2 + [w(a)]^2 + [w(b)]^2$.

Proof Since f, w are relative logarithmic semi-convex functions, then we have

$$\begin{aligned}
&\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(a)+g(b)-g(x)\bigr)\,dg(x)\\
&\quad=\int_0^1 f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl((1-t)g(a)+tg(b)\bigr)\,dt\\
&\quad\le\int_0^1 \bigl[f(a)\bigr]^t\bigl[f(b)\bigr]^{1-t}\bigl[w(a)\bigr]^{1-t}\bigl[w(b)\bigr]^t\,dt\\
&\quad=\frac{f(a)w(b)-f(b)w(a)}{\log f(a)w(b)-\log f(b)w(a)}\\
&\quad=L\bigl[f(a)w(b),f(b)w(a)\bigr]\le\frac{f(a)w(b)+f(b)w(a)}{2}\\
&\quad\le\frac{1}{2}\int_0^1\bigl\{\bigl[f\bigl(tg(a)+(1-t)g(b)\bigr)\bigr]^2+\bigl[w\bigl((1-t)g(a)+tg(b)\bigr)\bigr]^2\bigr\}\,dt\\
&\quad\le\frac{1}{2}\int_0^1\bigl\{\bigl[f(a)\bigr]^t\bigl[f(b)\bigr]^{1-t}\bigr\}^2\,dt+\frac{1}{2}\int_0^1\bigl\{\bigl[w(a)\bigr]^{1-t}\bigl[w(b)\bigr]^t\bigr\}^2\,dt\\
&\quad=\frac{1}{4}\frac{[f(a)]^2-[f(b)]^2}{\log f(a)-\log f(b)}+\frac{1}{4}\frac{[w(a)]^2-[w(b)]^2}{\log w(a)-\log w(b)}\\
&\quad=\frac{1}{2}\biggl[\frac{f(a)+f(b)}{2}\frac{f(a)-f(b)}{\log f(a)-\log f(b)}\biggr]+\frac{1}{2}\biggl[\frac{w(a)+w(b)}{2}\frac{w(a)-w(b)}{\log w(a)-\log w(b)}\biggr]\\
&\quad\le\frac{1}{2}\biggl[\frac{f(a)+f(b)}{2}\frac{f(a)+f(b)}{2}\biggr]+\frac{1}{2}\biggl[\frac{w(a)+w(b)}{2}\frac{w(a)+w(b)}{2}\biggr]\le\frac{1}{4}\Theta(a,b),
\end{aligned}$$

which is the required result. □

Theorem 3.20 *Let $f,w:I\to(0,\infty)$ be increasing and relative logarithmic semi-convex functions on I with $g(a),g(b)\in I$. Then we have*

$$\begin{aligned}
&f\biggl(\frac{g(a)+g(b)}{2}\biggr)L\bigl[w(a),w(b)\bigr]+w\biggl(\frac{g(a)+g(b)}{2}\biggr)L\bigl[f(a),f(b)\bigr]\\
&\quad\le\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\bigl(g(x)\bigr)w\bigl(g(x)\bigr)\,dg(x)+L\bigl[f(a)w(a),f(b)w(b)\bigr].
\end{aligned}$$

Proof Let f and w be relative logarithmic semi-convex functions. Then

$$\begin{aligned}
&f\bigl(tg(a)+(1-t)g(b)\bigr)\le\bigl[f(a)\bigr]^t\bigl[f(b)\bigr]^{1-t},\\
&w\bigl(tg(a)+(1-t)g(b)\bigr)\le\bigl[w(a)\bigr]^t\bigl[w(b)\bigr]^{1-t}.
\end{aligned}$$

Now, using $\langle x_1-x_2,x_3-x_4\rangle\ge 0$ $(x_1,x_2,x_3,x_4\in\mathbb{R})$ and $x_1<x_2<x_3<x_4$, we have

$$\begin{aligned}
&f\bigl(tg(a)+(1-t)g(b)\bigr)\bigl[w(a)\bigr]^t\bigl[w(b)\bigr]^{1-t}+w\bigl(tg(a)+(1-t)g(b)\bigr)\bigl[f(a)\bigr]^t\bigl[f(b)\bigr]^{1-t}\\
&\quad\le f\bigl(tg(a)+(1-t)g(b)\bigr)w\bigl(tg(a)+(1-t)g(b)\bigr)+\bigl[f(a)\bigr]^t\bigl[f(b)\bigr]^{1-t}\bigl[w(a)\bigr]^t\bigl[w(b)\bigr]^{1-t}.
\end{aligned}$$

Integrating the above inequalities with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f\big(tg(a)+(1-t)g(b)\big)\big[w(a)\big]^t\big[w(b)\big]^{1-t}\,dt \\
&\qquad+\int_0^1 w\big(tg(a)+(1-t)g(b)\big)\big[f(a)\big]^t\big[f(b)\big]^{1-t}\,dt \\
&\quad\le\int_0^1 f\big(tg(a)+(1-t)g(b)\big)w\big(tg(a)+(1-t)g(b)\big)\,dt \\
&\qquad+\int_0^1 \big[f(a)\big]^t\big[f(b)\big]^{1-t}\big[w(a)\big]^t\big[w(b)\big]^{1-t}\,dt.
\end{aligned}$$

Now, since f and w are increasing, using Chebyshev inequalities [28], we have

$$\begin{aligned}
&\int_0^1 f\big(tg(a)+(1-t)g(b)\big)\,dt\int_0^1\big[w(a)\big]^t\big[w(b)\big]^{1-t}\,dt \\
&\qquad+\int_0^1 w\big(tg(a)+(1-t)g(b)\big)\,dt\int_0^1\big[f(a)\big]^t\big[f(b)\big]^{1-t}\,dt \\
&\quad\le\int_0^1 f\big(tg(a)+(1-t)g(b)\big)w\big(tg(a)+(1-t)g(b)\big)\,dt \\
&\qquad+\int_0^1 \big[f(a)\big]^t\big[f(b)\big]^{1-t}\big[w(a)\big]^t\big[w(b)\big]^{1-t}\,dt.
\end{aligned}$$

Now calculating the simple integration, we have

$$\begin{aligned}
&\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)\,dg(x)L\big[w(a),w(b)\big] \\
&\qquad+\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} w\big(g(x)\big)\,dg(x)L\big[f(a),f(b)\big] \\
&\quad\le\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f\big(g(x)\big)w\big(g(x)\big)\,dg(x)+L\big[f(a)w(a),f(b),w(b)\big].
\end{aligned}$$

Now, using the left-hand side of Hermite-Hadamard's inequality for relative logarithmic semi-convex functions, we have the required result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MAN, MUA and KIN worked jointly. All the authors read and approved the final manuscript.

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